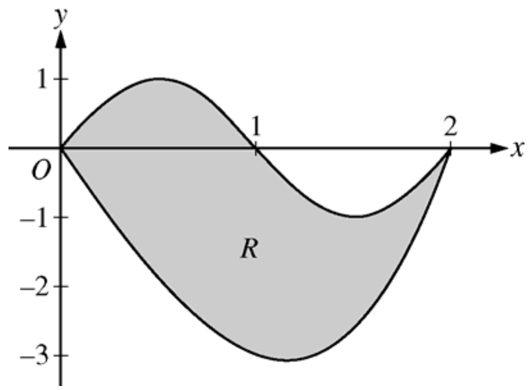
AP® CALCULUS AB
2008 SCORING GUIDELINES

Question 1



Let R be the region bounded by the graphs of $y = \sin(\pi x)$ and $y = x^3 - 4x$, as shown in the figure above.

(a) Find the area of R.

(b) The horizontal line $y = -2$ splits the region R into two parts. Write, but do not evaluate, an integral expression for the area of the part of R that is below this horizontal line.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

(d) The region R models the surface of a small pond. At all points in R at a distance x from the y-axis, the depth of the water is given by $h(x) = 3 - x$. Find the volume of water in the pond.

(a) $\sin(\pi x) = x^3 - 4x$ at $x = 0$ and $x = 2$ Area $= \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 4$	$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(b) $x^3 - 4x = -2$ at $r = 0.5391889$ and $s = 1.6751309$ The area of the stated region is $\int_r^s \left(-2 - \left(x^3 - 4x\right)\right) dx$	$2 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \end{cases}$
(c) Volume $= \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right)^2 dx = 9.978$	$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(d) Volume $= \int_0^2 (3 - x)\left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 8.369$ or 8.370	$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

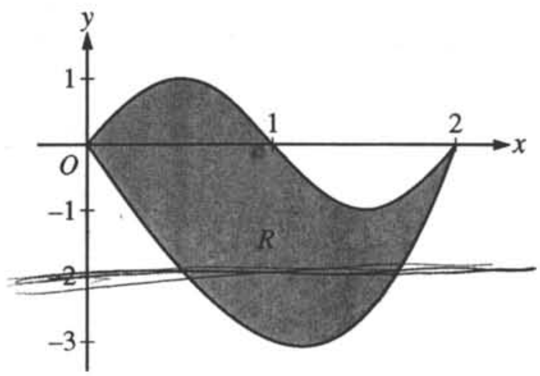
CALCULUS BC

SECTION II, Part A

Time—45 minutes

Number of problems—3

A graphing calculator is required for some problems or parts of problems.



Work for problem 1(a)

$\sin(\pi x) = x^3 - 4x$

$x = 2$

$A = \int_0^2 \sin(\pi x) - (x^3 - 4x)\,dx$

$A = 4$

Work for problem 1(b)

$-2 = x^3 - 4x$

$x = .53918887$, and 1.6751309

$A = \int_{.539187}^{1.6751309} (-2) - (x^3 - 4x)\,dx$

$= \int_{.5391887}^{1.6751309} -2 - x^3 + 4x\,dx$

Do not write beyond this border.

Continue problem 1 on page 5

Work for problem 1(c)

$$V = \int_0^2 \left(\sin(\pi x) - (x^3 - 4x)\right)^2 dx$$

$V = 9.978344126$

Work for problem 1(d)

$$V = \int_0^2 \left(\sin(\pi x) - (x^3 - 4x)\right)(3 - x)\, dx$$

$V = 8.369953106$

Do not write beyond this border.

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

1B. 1 1 1 1 1 1 1 1 1 1 1B

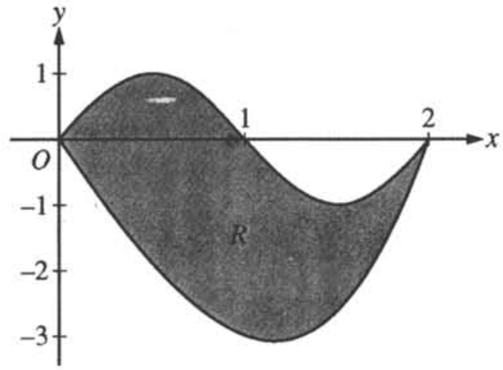
CALCULUS AB

SECTION II, Part A

Time—45 minutes

Number of problems—3

A graphing calculator is required for some problems or parts of problems.



Work for problem 1(a)

$A = \sin(\pi x) - (x^3 - 4x)$

$\sin(\pi x) = x^3 - 4x$

$x = -2 \quad x = 0 \quad x = 2$

$\int_0^2 [\sin(\pi x) - (x^3 - 4x)]\,dx = 4$

Work for problem 1(b)

$\sin(\pi x) - (x^3 - 4x)$

$y = -2$

$[\sin(\pi x) - \underset{-x^3+4x}{(x^3 - 4x)}] - (-2)$

$\int_0^2 ([(\sin(\pi x) - x^3 + 4x)] + 2)\,dx$

Do not write beyond this border.

Continue problem 1 on page 5.

Do not write beyond this border.

Work for problem 1(c)

cross section = square

$A = s^2$

$= [\sin \pi x - x^3 + 4x]^2$

$V = \int_0^2 [\sin \pi x - x^3 + 4x]^2 dx = 9.9783$

Work for problem 1(d)

$h(x) = 3 - x$ (depth)

$V = \pi \int_0^2 [\sin \pi x - x^3 + 4x](3-x)\, dx$

26.2950

Do not write beyond this border.

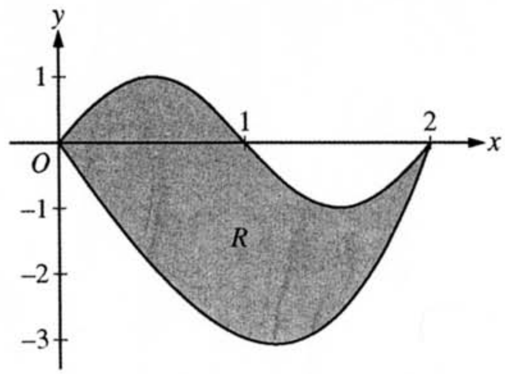
CALCULUS AB

SECTION II, Part A

Time—45 minutes

Number of problems—3

A graphing calculator is required for some problems or parts of problems.



Work for problem 1(a)

$$\int_0^2 (\sin(\pi x)) - (x^3 - 4x)\, dx = 4$$

Work for problem 1(b)

$$\int_0^2 (-2) - (x^3 - 4x)\, dx$$

Do not write beyond this border.

Do not write beyond this border.

Continue problem 1 on page 5.

Work for problem 1(c)

$$V_{\text{above}} = \int_0^2 (\sin(\pi x))^2 - (x^3-4x)^2 = 8.752$$

Work for problem 1(d)

$$V_{\text{water in pond}} = 4\pi \int_0^2 \frac{1}{2}(3-x)^2 = 17.333\pi$$

Do not write beyond this border.

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

Question 1

Overview

In this problem, students were given the graph of a region R bounded by two curves in the xy-plane. The points of intersection of the two curves were observable from the supplied graph. The formulas for the curves were given—a trigonometric function and a cubic polynomial—and students needed to match the appropriate functions to the upper and lower bounding curves. In each part, students had to set up and evaluate an appropriate integral. Part (a) asked for the area of R. Part (b) asked for the area of the portion of R below the line $y = -2,$ so students needed to use a calculator to solve for the x-coordinates of the points of intersection of $y = -2$ and the lower curve to set up the appropriate integral. Part (c) asked for the volume of a solid with base R whose cross sections perpendicular to the x-axis are squares. In part (d) students were asked to find a volume in an applied setting. They had to determine that cross sections perpendicular to the x-axis are rectangles with one dimension in region R and the other dimension supplied by $h(x) = 3 - x.$

Sample: 1A
Score: 9

The student earned all 9 points. In part (a) the student has the correct limits, integrand, and answer and earned all 3 points. In part (b) the limits are correct to three decimal places, and the limits point was earned. The student earned the integrand point on the first presentation of the integral. The second, simplified integral is also correct. In part (c) the student earned the integrand point with a correct integrand and earned the answer point since the answer is correct to three decimal places. In part (d) the integrand is correct, and the answer is correct to three decimal places.

Sample: 1B
Score: 6

The student earned 6 points: 3 points in part (a), no points in part (b), 2 points in part (c), and 1 point in part (d). In part (a) the student has the correct limits, integrand, and answer and earned all 3 points. In part (b) the student does not find the intersection of the cubic curve with the line $y = -2,$ and so the limits point was not earned. The integrand is not correct. In part (c) the integrand and answer are correct. The student earned both points. In part (d), although the integrand is correct, the student multiplies the integral by $\pi,$ and so the answer point was not earned.

Sample: 1C
Score: 4

The student earned 4 points: 3 points in part (a), 1 point in part (b), no points in part (c), and no points in part (d). In part (a) the student has the correct limits, integrand, and answer and earned all 3 points. In part (b) the student earned the integrand point. The student does not find the intersection of the cubic curve with the line $y = -2,$ and so the limits point was not earned. In part (c) the student integrates a difference of squares, rather than the square of the difference of the functions. The integrand point was not earned, and the student was not eligible for the answer point. In part (d) the integrand is not correct, and the student was not eligible for the answer point.

Question 2

t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

Concert tickets went on sale at noon $(t = 0)$ and were sold out within 9 hours. The number of people waiting in line to purchase tickets at time t is modeled by a twice-differentiable function L for $0 \le t \le 9$. Values of $L(t)$ at various times t are shown in the table above.

(a) Use the data in the table to estimate the rate at which the number of people waiting in line was changing at 5:30 P.M. $(t = 5.5)$. Show the computations that lead to your answer. Indicate units of measure.

(b) Use a trapezoidal sum with three subintervals to estimate the average number of people waiting in line during the first 4 hours that tickets were on sale.

(c) For $0 \le t \le 9$, what is the fewest number of times at which $L'(t)$ must equal 0 ? Give a reason for your answer.

(d) The rate at which tickets were sold for $0 \le t \le 9$ is modeled by $r(t) = 550te^{-t/2}$ tickets per hour. Based on the model, how many tickets were sold by 3 P.M. $(t = 3)$, to the nearest whole number?

(a) $L'(5.5) \approx \dfrac{L(7) - L(4)}{7 - 4} = \dfrac{150 - 126}{3} = 8$ people per hour

$2 : \begin{cases} 1 : \text{estimate} \\ 1 : \text{units} \end{cases}$

(b) The average number of people waiting in line during the first 4 hours is approximately

$$\frac{1}{4}\left(\frac{L(0) + L(1)}{2}(1 - 0) + \frac{L(1) + L(3)}{2}(3 - 1) + \frac{L(3) + L(4)}{2}(4 - 3)\right)$$

$= 155.25$ people

$2 : \begin{cases} 1 : \text{trapezoidal sum} \\ 1 : \text{answer} \end{cases}$

(c) L is differentiable on $[0, 9]$ so the Mean Value Theorem implies $L'(t) > 0$ for some t in $(1, 3)$ and some t in $(4, 7)$. Similarly, $L'(t) < 0$ for some t in $(3, 4)$ and some t in $(7, 8)$. Then, since L' is continuous on $[0, 9]$, the Intermediate Value Theorem implies that $L'(t) = 0$ for at least three values of t in $[0, 9]$.

$3 : \begin{cases} 1 : \text{considers change in sign of } L' \\ 1 : \text{analysis} \\ 1 : \text{conclusion} \end{cases}$

OR

The continuity of L on $[1, 4]$ implies that L attains a maximum value there. Since $L(3) > L(1)$ and $L(3) > L(4)$, this maximum occurs on $(1, 4)$. Similarly, L attains a minimum on $(3, 7)$ and a maximum on $(4, 8)$. L is differentiable, so $L'(t) = 0$ at each relative extreme point on $(0, 9)$. Therefore $L'(t) = 0$ for at least three values of t in $[0, 9]$.

OR

$3 : \begin{cases} 1 : \text{considers relative extrema of } L \text{ on } (0, 9) \\ 1 : \text{analysis} \\ 1 : \text{conclusion} \end{cases}$

[*Note: There is a function* L *that satisfies the given conditions with* $L'(t) = 0$ *for exactly three values of* t.]

(d) $\int_0^3 r(t)\, dt = 972.784$

There were approximately 973 tickets sold by 3 P.M.

$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and answer} \end{cases}$

2A.

2 2 2 2 2

t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

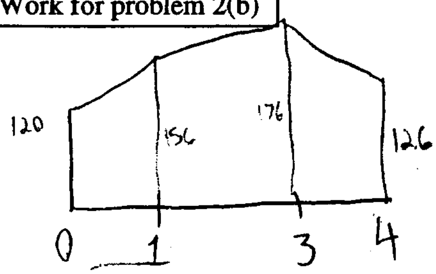
Work for problem 2(a)

$$L'(5.5) \approx \frac{L(7)-L(4)}{7-4} = \frac{150-126}{3} = \frac{24}{3} = 8$$

8 people in line per hour

Do not write beyond this border.

Work for problem 2(b)



$$\frac{120+156}{2} + 156+176 + \frac{176+126}{2} =$$

$\downarrow$

$138+156+176+151 = 621$

average people in line $= \dfrac{621}{4} = $ 155.25 people

Work for problem 2(c) Since $L(3) > L(1)$ and $L(3) > L(4)$, at some point between $t=1$ and $t=4$, the line must go from increasing to decreasing, and thus at some point $L'(t)=0$

Since $L(4) < L(3)$ and $L(4) < L7$, there must be a local minimum between $t=3$ and $t=7$, and thus another point where $L'(t)=0$

Since $L(7) > L(4)$ and $L(7) > L(8)$, there must be another local maximum between $t=4$ and $t=8$ and thus a point where $L'(t)=0$

There must be at least 3 points

Work for problem 2(d)

tickets sold $= \int_0^3 550te^{-\frac{t}{2}}\,dx = 972.784$

$\downarrow$

973

Do not write beyond this border.

Do not write beyond this border.

t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

2B1

Work for problem 2(a)

$$L'(t) \approx \frac{L(7)-L(4) \text{ people}}{7-4 \text{ hour}} = \frac{150-126 \text{ people}}{3 \text{ hour}} = \frac{24 \text{ people}}{3 \text{ hour}} = \boxed{8 \frac{\text{People}}{\text{hour}}}$$

Work for problem 2(b)

$$\text{Average} = \frac{1}{b-a}\left(\frac{b-a}{2n}\right)\left(L(0) + 2L(1) + 2L(3) + L(4)\right)$$

$$= \frac{1}{6}\left(120 + 312 + 352 + 126\right) = \frac{910}{6} = \boxed{\frac{455}{3} \text{ People}}$$

Do not write beyond this border.

Continue problem 2 on page 7.

Work for problem 2(c)

t	0	1	3	4	7	8	9
L(t)	120	156	176	126	150	80	0

L'(t) must equal zero at least 3 times

because must equal zero whenever L(t) changes from increasing to decreasing, or ~~from~~ decreasing to increasing, which we can only be sure of happening (based on the chart) between t=3 and t=4; between t=4 and t=7; and between t=7 and t=8.

Work for problem 2(d)

Tickets Sold $= \int_0^3 550te^{-t/2}\,dt \approx \boxed{973 \text{ tickets}}$

Do not write beyond this border.

2C1

t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

Do not write beyond this border.

Work for problem 2(a)

$$\frac{f(7) - f(4)}{7-4} = \frac{150-126}{3} = 8 \ \frac{\text{people}}{\text{hr}}$$

Work for problem 2(b)

$$\frac{120+156}{2} + \frac{156+176}{4} + \frac{176+126}{2} = 372$$

Do not write beyond this border.

Continue problem 2 on page 7.

χc^2

Work for problem 2(c)

2 times

Work for problem 2(d)

$r(t) = 550\, te^{-\frac{t}{2}}$ tickets/hour

$\int_0^3 550te^{-\frac{t}{2}} = 274.966$

≈ 275 tickets

Do not write beyond this border.

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

Question 2

Overview

This problem presented students with a table of data indicating the number of people $L(t)$ in line at a concert ticket office, sampled at seven times t during the 9 hours that tickets were being sold. (The question stated that $L(t)$ was twice differentiable.) Part (a) asked for an estimate for the rate of change of the number of people in line at a time that fell between the times sampled in the table. Students were to use data from the table to calculate an average rate of change to approximate this value. Part (b) asked for an estimate of the average number of people waiting in line during the first 4 hours and specified the use of a trapezoidal sum. Students needed to recognize that the computation of an average value involves a definite integral, approximate this integral with a trapezoidal sum, and then divide this total accumulation of people hours by 4 hours to obtain the average. Part (c) asked for the minimum number of solutions guaranteed for $L'(t) = 0$ during the 9 hours. Students were expected to recognize that a change in direction (increasing/decreasing) for a twice-differentiable function forces a value of 0 for its derivative. Part (d) provided the function $r(t) = 550te^{-t/2}$ tickets per hour as a model of the rate at which tickets were sold during the 9 hours and asked students to find the number of tickets sold in the first 3 hours, to the nearest whole number, using this model. Students needed to recognize that total tickets sold could be determined by a definite integral of the rate $r(t)$ at which tickets were sold.

Sample: 2A
Score: 9

The student earned all 9 points. In part (c) the student might have given a more complete justification for the existence of a local maximum on the interval $(1, 4)$. It would have been better if the student had used the word "graph" rather than "line" in the first paragraph. The response is more complete and uses better terminology in the second and third paragraphs, and the student gives the correct answer.

Sample: 2B
Score: 6

The student earned 6 points: 2 points in part (a), no points in part (b), 2 points in part (c), and 2 points in part (d). The student earned both points in part (a) with a correct estimate and correct units. In part (b) the student's expression reflects subdivisions of $[0, 4]$ of equal length. The student did not earn any points. In part (c) the student earned the first point by considering $L(t)$ changing from increasing to decreasing. (The student goes on to consider $L(t)$ changing from decreasing to increasing, but this was not necessary to earn the first point.) The student did not earn the second point since it is not necessarily true that $L(t)$ changes from increasing to decreasing on the interval $[3, 4]$. The student earned the third point with the correct answer of 3. The student earned both points in part (d).

Question 2 (continued)

Sample: 2C
Score: 3

The student earned 3 points: 2 points in part (a), no points in part (b), no points in part (c), and 1 point in part (d). The student earned both points in part (a) with a correct estimate and correct units. The student's use of f rather than L was not penalized. The student did not earn points in part (b) because the given expression is not a valid trapezoidal sum. In part (c) the student did not earn the first point. As a result of the incorrect answer ("two times"), the student did not earn the other points in part (c). In part (d) the student earned the integrand point. Since the value of the integral is not correct, the student did not earn the second point, even though the limits on the integral are correct.

Question 3

Oil is leaking from a pipeline on the surface of a lake and forms an oil slick whose volume increases at a constant rate of 2000 cubic centimeters per minute. The oil slick takes the form of a right circular cylinder with both its radius and height changing with time. (Note: The volume V of a right circular cylinder with radius r and height h is given by $V = \pi r^2 h$.)

(a) At the instant when the radius of the oil slick is 100 centimeters and the height is 0.5 centimeter, the radius is increasing at the rate of 2.5 centimeters per minute. At this instant, what is the rate of change of the height of the oil slick with respect to time, in centimeters per minute?

(b) A recovery device arrives on the scene and begins removing oil. The rate at which oil is removed is $R(t) = 400\sqrt{t}$ cubic centimeters per minute, where t is the time in minutes since the device began working. Oil continues to leak at the rate of 2000 cubic centimeters per minute. Find the time t when the oil slick reaches its maximum volume. Justify your answer.

(c) By the time the recovery device began removing oil, 60,000 cubic centimeters of oil had already leaked. Write, but do not evaluate, an expression involving an integral that gives the volume of oil at the time found in part (b).

(a) When $r = 100$ cm and $h = 0.5$ cm, $\frac{dV}{dt} = 2000 \text{ cm}^3/\text{min}$ and $\frac{dr}{dt} = 2.5 \text{ cm/min}$.

$$\frac{dV}{dt} = 2\pi r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt}$$

$$2000 = 2\pi(100)(2.5)(0.5) + \pi(100)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = 0.038 \text{ or } 0.039 \text{ cm/min}$$

$4: \begin{cases} 1: \frac{dV}{dt} = 2000 \text{ and } \frac{dr}{dt} = 2.5 \\ 2: \text{expression for } \frac{dV}{dt} \\ 1: \text{answer} \end{cases}$

(b) $\frac{dV}{dt} = 2000 - R(t)$, so $\frac{dV}{dt} = 0$ when $R(t) = 2000$.
This occurs when $t = 25$ minutes.
Since $\frac{dV}{dt} > 0$ for $0 < t < 25$ and $\frac{dV}{dt} < 0$ for $t > 25$, the oil slick reaches its maximum volume 25 minutes after the device begins working.

$3: \begin{cases} 1: R(t) = 2000 \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$

(c) The volume of oil, in cm^3, in the slick at time $t = 25$ minutes is given by $60{,}000 + \int_0^{25} (2000 - R(t))\, dt$.

$2: \begin{cases} 1: \text{limits and initial condition} \\ 1: \text{integrand} \end{cases}$

Work for problem 3(a)

$\frac{dV}{dt} = 2000 \frac{cm^3}{min}$ $\frac{dr}{dt} = 2.5 \frac{cm}{min}$ $\frac{dh}{dt} = ?$

$r = 100$ cm $h = .5$ cm

$V = \pi r^2 h$

$\frac{dV}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt} \cdot h + \pi r^2 \cdot \frac{dh}{dt}$

$2000 = \pi\left(2 \cdot 100 \cdot 2.5 \cdot .5 + (100)^2 \cdot \frac{dh}{dt}\right)$

$\frac{2000}{\pi} - 250 = 100^2 \cdot \frac{dh}{dt}$

$\frac{dh}{dt} = .039 \frac{cm}{min}$

Do not write beyond this border.

Continue problem 3 on page 9.

3A2

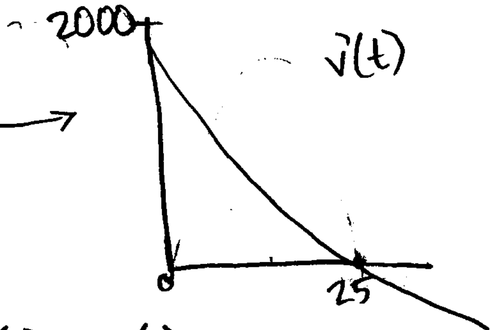
Work for problem 3(b)

$$V(t)=\int_0^t 2000\cdot dt-\int_0^t 400\sqrt{t}\cdot dt$$

$$v'(t)=2000-400\sqrt{t}\ \cancel{=2000t-\frac{800}{3}t^{3/2}}$$

$$0=2000-400\sqrt{t}$$

$$t=25\text{ min}$$



*point where $v'(t)$ changes from (+) to (-) is where $v(t)$ has a local maximum

$\rightarrow v''(t)=-\frac{200}{\sqrt{t}}$ always negative when $t>0 \rightarrow v(t)$ always concave down making the local max. of $t=25$ a global maximum

Work for problem 3(c)

*$t=0$ is when device began working

$$\text{Volume}=\int_0^{25} 2000-400\sqrt{t}\cdot dt+60{,}000$$

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

Do not write beyond this border.

Do not write beyond this border.

3B1

Work for problem 3(a)

$\frac{dV}{dt} = 2000 \text{ cm}^3/\text{min}$

$V = \pi r^2 h$

when $r = 100$ $h = 0.5$

$\frac{dr}{dt} = 2.5$

find: $\frac{dh}{dt}$!

$\frac{dV}{dt} = 2\pi r \frac{dr}{dt} \cdot h + \pi r^2 \cdot \frac{dh}{dt}$

$2000 = 2 \cdot \pi \cdot 100 \cdot 0.5 \cdot 2.5 + \pi \cdot 10000 \cdot \frac{dh}{dt}$

$2000 - 785.398 = 10000\pi \cdot \frac{dh}{dt}$

$1214.602 = 10000\pi \cdot \frac{dh}{dt}$

$0.387 = \frac{dh}{dt}$

$\frac{dh}{dt}$ at the given instant is 0.387 cm/min

↳ the height of the oil slick is changing at that rate

Do not write beyond this border.

Do not write beyond this border.

Continue problem 3 on page 9.

3B2

Work for problem 3(b)

$R(t) = 400\sqrt{t}$ cm^3/min

Volume of Oil present = $\cancel{\int 2000 - 400\sqrt{t}}$

$V = 2000 - 400\sqrt{t}$

$2000 - 400\sqrt{t} = 0$

$t = 25$

after 25 minutes the Volume reaches its maximum value.

Work for problem 3(c)

$V = 6000 - \int_0^{25} (2000 - 400\sqrt{t})\,dt$ = volume of oil at the time $t = 25$

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

Do not write beyond this border.

3C1

-Do not write beyond this border.

Work for problem 3(a)

$\frac{dv}{dt} = 2000 \text{cm}^3/\text{min}$ $\qquad V = \pi r^2 h$

$r = 100 \quad h = .5 \quad \frac{dr}{dt} = 2.5 \text{cm}/\text{min}$

$\frac{dv}{dt} = \pi 2r \frac{dr}{dt} \frac{dh}{dt}$

$2000 = 2\pi (100)(2.5) \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{4}{\pi} \text{cm}/\text{min}$ = rate of change of height

Continue problem 3 on page 9

3C2

Work for problem 3(b)

rate increases by

$\frac{dv}{dt} = 2000$ cm/min

$2000 - 400\sqrt{t} = 0$

@ $t = \sqrt{5}$

~~The oil slick reaches its maximum volume @ t = 0 because the amount of oil being removed is least when t=0) Since the oil is leaking at a constant rate, the factor that is important is how much is being removed, and that increases as times goes by, as shown by R(t)~~

The oil slick reaches its max volume @ $t = \sqrt{5}$ because after $t = \sqrt{5}$, the recovery device begins pumping more than 2000 cm³ of oil out of the lake, so the rate at which oil was entering the pond went from increasing to decrease ∴ a max was attained

Work for problem 3(c)

$$\text{Oil} = 60000 + \int_0^{\sqrt{5}} 2000 - R(t)\,dt$$

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

Do not write beyond this border.

Question 3

Overview

This problem presented students with a scenario in which oil leaking from a pipeline into a lake organizes itself as a dynamic cylinder whose height and radius change with time. The rate at which oil is leaking into the lake was given as 2000 cubic centimeters per minute. Part (a) was a related-rates problem; students needed to use the chain rule to differentiate volume, $V = \pi r^2 h$, with respect to time and determine the rate of change of the oil slick's height at an instant when the oil slick has radius 100 cm and height 0.5 cm, and its radius is increasing at 2.5 cm/min. In part (b) an oil recovery device arrives on the scene; as the pipeline continues to leak at 2000 cubic centimeters per minute, the device removes oil at the rate of $R(t) = 400\sqrt{t}$ cubic centimeters per minute, with t measured in minutes from the time the device began removing oil. Students were asked for the time t when the volume of the oil cylinder is greatest. They needed to recognize the rate of change of the volume of oil in the lake, $\frac{dV}{dt}$, as the difference between the rate at which oil enters the lake from the leak and the rate at which it is removed by the device. A sign analysis of $\frac{dV}{dt}$ or an application of the Second Derivative Test and the critical point theorem could justify that the critical point found yields a maximum value for the volume of the oil cylinder. Part (c) tested students' ability to use the Fundamental Theorem of Calculus to find the amount of oil in the lake at the time found in part (b), given that 60,000 cubic centimeters had already leaked when the recovery device began its task.

Sample: 3A
Score: 9

The student earned all 9 points.

Sample: 3B
Score: 6

The student earned 6 points: 3 points in part (a), 2 points in part (b), and 1 point in part (c). In part (a) the student correctly notes that $\frac{dV}{dt} = 2000$ and $\frac{dr}{dt} = 2.5$ and gives the correct symbolic expression for $\frac{dV}{dt}$. The student earned the first 3 points. The student makes an error in calculating the numerical value for $\frac{dh}{dt}$ so did not earn the last point. In part (b) the student earned the first 2 points for solving $2000 - 400\sqrt{t} = 0$ to find $t = 25$ as the time when the volume reaches a maximum. The student provides no justification and did not earn the third point. In part (c) the student uses the correct initial condition and the correct limits of integration, earning the first point. The integrand is the negative of what is needed to calculate the volume, so the integrand point was not earned.

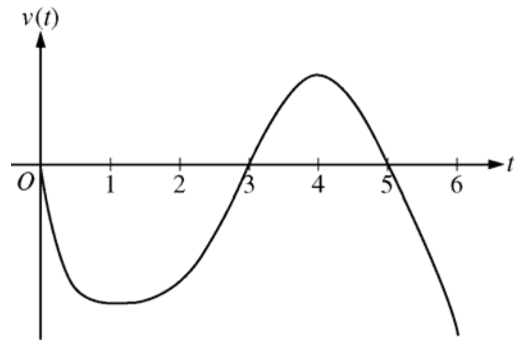
Question 3 (continued)

Sample: 3C
Score: 4

The student earned 4 points: 1 point in part (a), 1 point in part (b), and 2 points in part (c). In part (a) the student correctly notes that $\frac{dV}{dt} = 2000$ and $\frac{dr}{dt} = 2.5$ but does not use the product rule correctly. The student did not earn either derivative point and was not eligible for the answer point. In part (b) the student solves $2000 - 400\sqrt{t} = 0$ incorrectly, which earned the first point but not the second one. The student did not earn the third point. The justification is not correct for the student's value of t and is only a local argument where a global argument is required. In part (c) the student earned both points. The initial condition is correct, and the student is allowed to import the incorrect value of t found in part (b) as the upper limit of integration. The integrand is correct.

AP® CALCULUS AB
2008 SCORING GUIDELINES

Question 4



Graph of v

A particle moves along the x-axis so that its velocity at time t, for $0 \le t \le 6$, is given by a differentiable function v whose graph is shown above. The velocity is 0 at $t = 0$, $t = 3$, and $t = 5$, and the graph has horizontal tangents at $t = 1$ and $t = 4$. The areas of the regions bounded by the t-axis and the graph of v on the intervals $[0, 3]$, $[3, 5]$, and $[5, 6]$ are 8, 3, and 2, respectively. At time $t = 0$, the particle is at $x = -2$.

(a) For $0 \le t \le 6$, find both the time and the position of the particle when the particle is farthest to the left. Justify your answer.

(b) For how many values of t, where $0 \le t \le 6$, is the particle at $x = -8$? Explain your reasoning.

(c) On the interval $2 < t < 3$, is the speed of the particle increasing or decreasing? Give a reason for your answer.

(d) During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

(a) Since $v(t) < 0$ for $0 < t < 3$ and $5 < t < 6$, and $v(t) > 0$ for $3 < t < 5$, we consider $t = 3$ and $t = 6$.

$$x(3) = -2 + \int_0^3 v(t)\,dt = -2 - 8 = -10$$

$$x(6) = -2 + \int_0^6 v(t)\,dt = -2 - 8 + 3 - 2 = -9$$

Therefore, the particle is farthest left at time $t = 3$ when its position is $x(3) = -10$.

$3 : \begin{cases} 1 : \text{identifies } t = 3 \text{ as a candidate} \\ 1 : \text{considers } \int_0^6 v(t)\,dt \\ 1 : \text{conclusion} \end{cases}$

(b) The particle moves continuously and monotonically from $x(0) = -2$ to $x(3) = -10$. Similarly, the particle moves continuously and monotonically from $x(3) = -10$ to $x(5) = -7$ and also from $x(5) = -7$ to $x(6) = -9$.

By the Intermediate Value Theorem, there are three values of t for which the particle is at $x(t) = -8$.

$3 : \begin{cases} 1 : \text{positions at } t = 3,\ t = 5, \\ \quad\ \text{and } t = 6 \\ 1 : \text{description of motion} \\ 1 : \text{conclusion} \end{cases}$

(c) The speed is decreasing on the interval $2 < t < 3$ since on this interval $v < 0$ and v is increasing.

1 : answer with reason

(d) The acceleration is negative on the intervals $0 < t < 1$ and $4 < t < 6$ since velocity is decreasing on these intervals.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

4A

NO CALCULATOR ALLOWED

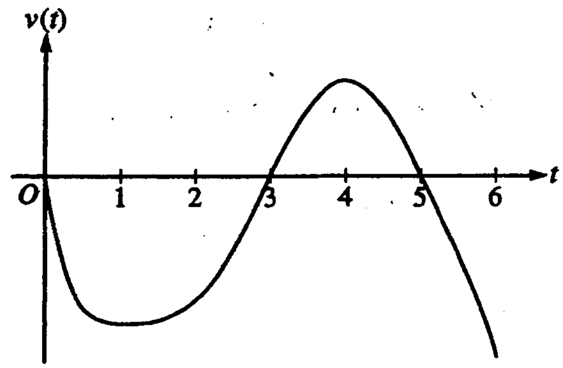
CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



Graph of v

Work for problem 4(a)

$x(t) = \int v(t)\,dt$

$x(0) = -2$

farthest to the left = greatest negative $x(t)$

from $[0,3]$ constantly moving to left

position: $x(0) - 8 = x(3)$

$-2 - 8 = x(3)$

$-10 = x(3)$

during $[3,5]$ moving to right

position: $x(0) - 8 + 3 = x(5)$ $\quad -2-8+3 = -7 = x(5)$

$[5,6]$ moving left

total displacement:

$x(0) - 8 + 3 - 2 =$

$-2 - 8 + 3 - 2 = -9$

farthest to the left @ $t=3$, where $x = -10$

Work for problem 4(b)

$[0,3]$ particle moves from -2 to -10 (passes $x=-8$ once here)

$[3,5]$ moves from -10 to -7 (passes $x=-8$ once here)

$[5,6]$ from -7 to -9 passes -8

3 times particle is at 8 b/c displacement calculations crossed $x=-8$ three times.

Continue problem 4 on page 11.

NO CALCULATOR ALLOWED

Work for problem 4(c)

the speed of the particle is decreasing because acceleration ($v'(t)$) and the direction of movement are in opposite directions. (acceleration is + but $v(t)$ is negative)

Work for problem 4(d)

Acceleration is negative from $[0,1) \vee (4,6]$ because the slope of velocity ($v'(t)$) is negative there.

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

4B1

NO CALCULATOR ALLOWED

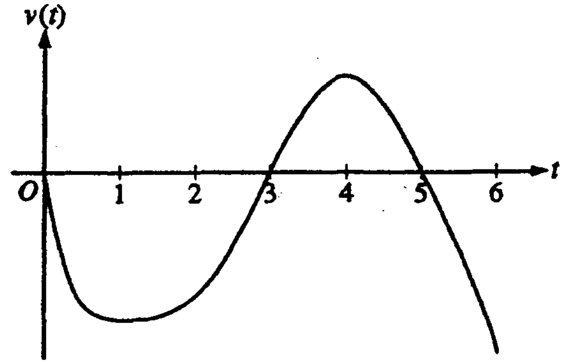
CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



Graph of v

Work for problem 4(a)

The point at which the particle is farthest to the left is $t=3$ here the particle would be at $x=-10$

This is so because the particle starts at position $x=-2$ then decreases on the graph for a total of 8 from $(0,3)$. The point farthest left could not be at any other point because after $t=3$ the velocity is positive so the particle is moving right. This positive value from $(3,5)$ has a greater area than $(5,6)$ so the particle ends farther to the right than at 3

Work for problem 4(b)

On the interval $0 \leq t \leq 6$ $x=-8$ three times

on the interval $(0,3)$ it adds -8 to the -2 so it passes -8 on some point and continues to -10 on $(3,5)$ it adds 3 bringing it to -7 passing -8 again and finally on $(5,6)$ it adds -2 bringing it to $x=-9$ passing -8 for the third time.

Do not write beyond this border.

Continue problem 4 on page 11.

NO CALCULATOR ALLOWED

Work for problem 4(c)

on the interval $2 \le t \le 3$ the speed of the particle is increasing this is because from (2,3) the graph has a positive slope and the graph of v is velocity so if it is a positive slope it has positive acceleration so the particle is increasing in speed.

Work for problem 4(d)

The acceleration of the particle is negative on the intervals $0 \le t \le .5$ and $4 \le t \le 6$ this is because the slope of the graph of velocity is negative at these points.

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

4 4 4 4 4 4C1

NO CALCULATOR ALLOWED

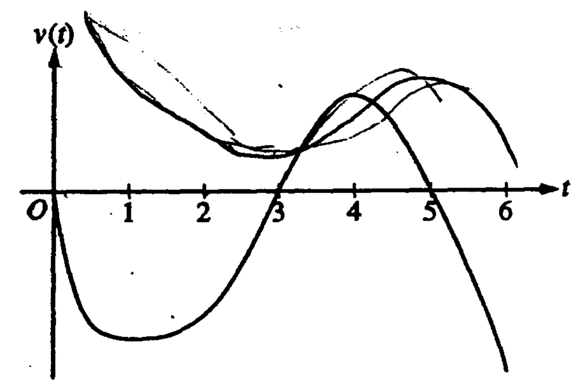
CALCULUS BC

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



Graph of v

Work for problem 4(a)

$\frac{dx}{dt} = v(t) = 0$ dx/dt

(−2, 0)

$\frac{dx}{dt} = 0$ ~~at t = 1, 4~~

at $\boxed{t=3}$, it is the farthest left b/c $\frac{dx}{dt}$ goes from negative to positive

position − 8 − 2

The position is at $\boxed{x = 6}$

Work for problem 4(b)

the particle is at $x = -8$ at t-value, $t = 3$

because $\int_0^x v(t)\,dt = -8$ only when $x = 3$

Do not write beyond this border.

Continue problem 4 on page 11.

NO CALCULATOR ALLOWED

Work for problem 4(c)

$s = |v(t)|$ the speed is increasing because the graph of $v(t)$ is increasing on the interval $(2,3)$ and speed is the absolute value of the velocity ($a(t) = v'(t)$ is pos. on this interval.)

Work for problem 4(d)

the acceleration is negative on $(0,1)$ and $(4,6)$ because the velocity is decreasing on these intervals

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

Question 4

Overview

This problem presented students with the graph of a velocity function for a particle in motion along the x-axis for $0 \le t \le 6$. Areas of regions between the velocity curve and the t-axis were also given. Part (a) asked for the time and position of the particle when it is farthest left, so students needed to know that velocity is the derivative of position, and they had to be able to determine positions at critical times from the particle's initial position and areas of regions bounded by the velocity curve and the t-axis. Part (b) tested knowledge of the Intermediate Value Theorem applied to information about the particle's position function derived from its initial position and the supplied graph of its derivative. Part (c) asked students to interpret information about the speed of the particle from the velocity graph: namely, that if velocity is negative but increasing, then its absolute value, speed, is decreasing. Part (d) asked for the time intervals over which acceleration is negative, so students had to recognize that acceleration is the derivative of velocity. The sign of acceleration can be read from the intervals of increase/decrease of the velocity function.

Sample: 4A
Score: 9

The student earned all 9 points.

Sample: 4B
Score: 6

The student earned 6 points: 3 points in part (a), 3 points in part (b), no points in part (c), and no points in part (d). In part (a) the student clearly uses $t = 3,$ describes the motion of the particle over each interval, and draws the correct conclusion. The student earned all 3 points. In part (b) the student finds the positions of the particle at the appropriate times, describes how the particle passes $x = -8$ on each interval, and draws the correct conclusion. The student earned all 3 points. In part (c) the student concludes that the speed of the particle is increasing so did not earn the point. In part (d) the student provides intervals that are not correct. Since the intervals do not have correct endpoints, the student did not earn any points.

Sample: 4C
Score: 3

The student earned 3 points: 1 point in part (a), no points in part (b), no points in part (c), and 2 points in part (d). In part (a) the student identifies $t = 3$ as a candidate but uses a local minimum justification. As a result, the student did not earn the last 2 points. In part (b) the student has a correct conclusion based on the work presented in part (a) but provides a reason that is not correct. The student did not earn the first point because the positions at $t = 5$ and $t = 6$ are not considered. The student does not describe the motion of the particle from position to position so did not earn the second or third points. In part (c) the student concludes that the speed of the particle is increasing so did not earn the point. In part (d) the student provides correct intervals and justification and earned both points.

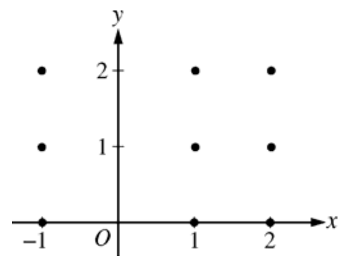
Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{y-1}{x^2}$, where $x \neq 0$.

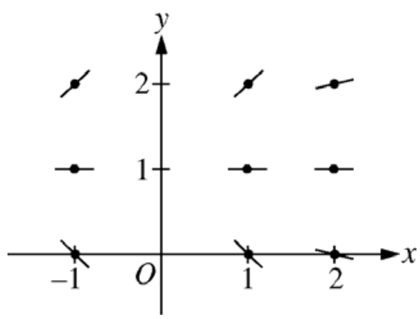
(a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.
(Note: Use the axes provided in the exam booklet.)

(b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = 0$.

(c) For the particular solution $y = f(x)$ described in part (b), find $\lim_{x\to\infty} f(x)$.



(a)



$2 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{all other slopes} \end{cases}$

(b) $\frac{1}{y-1}\,dy = \frac{1}{x^2}\,dx$

$\ln|y-1| = -\frac{1}{x} + C$

$|y-1| = e^{-\frac{1}{x}+C}$

$|y-1| = e^C e^{-\frac{1}{x}}$

$y - 1 = ke^{-\frac{1}{x}}$, where $k = \pm e^C$

$-1 = ke^{-\frac{1}{2}}$

$k = -e^{\frac{1}{2}}$

$f(x) = 1 - e^{\left(\frac{1}{2}-\frac{1}{x}\right)}$, $x > 0$

$6 : \begin{cases} 1 : \text{separates variables} \\ 2 : \text{antidifferentiates} \\ 1 : \text{includes constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

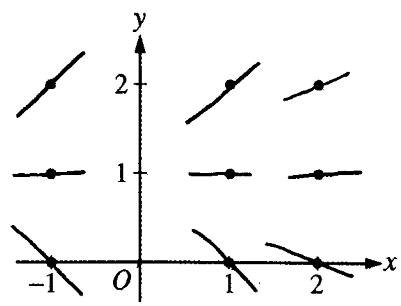
Note: max $3/6$ [1-2-0-0-0] if no constant of integration

Note: $0/6$ if no separation of variables

(c) $\lim_{x\to\infty} 1 - e^{\left(\frac{1}{2}-\frac{1}{x}\right)} = 1 - \sqrt{e}$

1 : limit

Work for problem 5(a)



$$\frac{dy}{dx} = \frac{y-1}{x^2}$$

$\frac{dy}{dx}$

(-1,0) : -1
(-1,1) : 0
(-1,2) : 1
(1,0) : -1
(1,1) : 0
(1,2) : 1
(2,0) : $-\frac{1}{4}$
(2,1) : 0
(2,2) : $\frac{1}{4}$

Do not write beyond this border.

Continue problem 5 on page 13.

5A2

Work for problem 5(b)

$\frac{dy}{dx} = \frac{y-1}{x^2}$

$\int \frac{dy}{y-1} = \int \frac{1}{x^2}\,dx$

$\int \frac{1}{y-1}\,dy = \int x^{-2}\,dx$

$\ln|y-1| = -x^{-1} + C$

$e^{\ln|y-1|} = e^{-x^{-1}+C}$

$y-1 = Ce^{-x^{-1}}$

$y = Ce^{-x^{-1}} + 1$

$0 = Ce^{-2^{-1}} + 1$

$0 = Ce^{-\frac{1}{2}} + 1$

$-1 = Ce^{-1/2}$

$-1 = \frac{C}{\sqrt{e}}$

$-\sqrt{e} = C$

$y = -\sqrt{e}\, e^{-x^{-1}} + 1$

Work for problem 5(c)

$\lim_{x\to\infty} -\sqrt{e}\, e^{-x^{-1}} + 1 = \lim_{x\to\infty} \frac{-\sqrt{e}}{e^{\frac{1}{x}}} + 1$

$\lim_{x\to\infty} \frac{-\sqrt{e}}{e^{\frac{1}{\infty}}} + 1$

0

$\lim_{x\to\infty} \frac{-\sqrt{e}}{1} + 1$

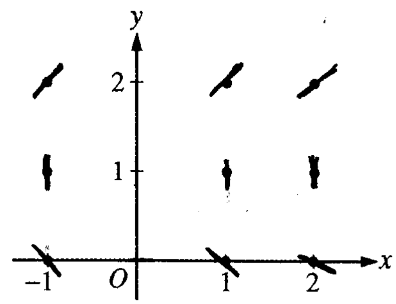
$\lim_{x\to\infty} f(x) = -\sqrt{e} + 1$

5B1

Do not write beyond this border.

Work for problem 5(a)

$$\frac{dy}{dx} = \frac{y-1}{x^2}$$



Continue problem 5 on page 13.

NO CALCULATOR ALLOWED

5B2

Do not write beyond this border.

Do not write beyond this border.

Work for problem 5(b)

$$\frac{dy}{dx} = \frac{y-1}{x^2}$$

$$\int \frac{1}{y-1}\,dy = \int \frac{1}{x^2}\,dx$$

$$\ln|y-1| = -x^{-1} + C$$

$$\ln|0-1| = -(2)^{-1} + C$$

$$0 = \frac{-1}{2} + C$$

$$C = \frac{1}{2}$$

$$y = \frac{1}{2}(x-2)$$

Work for problem 5(c)

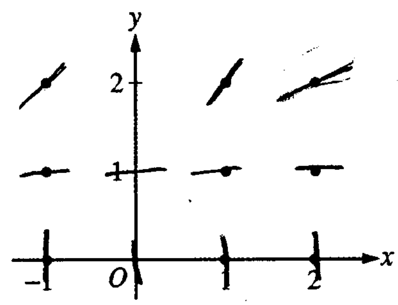
$$\lim_{x \to \infty} f(x) = \frac{1}{2}$$

GO ON TO THE NEXT PAGE.

NO CALCULATOR ALLOWED

5C1

Work for problem 5(a)



see graph

Do not write beyond this border.

Do not write beyond this border.

Continue problem 5 on page 13.

NO CALCULATOR ALLOWED

5C2

Work for problem 5(b)

$$\frac{dy}{dx} = \frac{y-1}{x^2}$$

$$\int \frac{dy}{y-1} = \int \frac{dx}{x^2} = \ln(y-1) = \ln x^2 + c$$

$$y - 1 = e^{\ln(x^2 + c)}$$

$$y = Ce^{\ln x^2} + 1$$

$$0 = Ce^{\ln 4} + 1$$

$$Ce^{\ln 4} = -1$$

$$C = -\frac{1}{e^{\ln 4}}$$

$$y = -\frac{1}{e^{\ln 4}} e^{\ln x^2} + 1$$

Do not write beyond this border.

Do not write beyond this border.

Work for problem 5(c)

this will keep getting bigger

$$\lim_{x \to \infty} -\frac{e^{\ln x^2}}{e^{\ln 4}} + 1 = -\infty$$

$$\frac{-\infty}{+\infty} + 1$$

GO ON TO THE NEXT PAGE.

Question 5

Overview

This problem presented students with a separable differential equation. In part (a) they were asked to sketch its slope field at nine sample points. Part (b) asked for the solution to the differential equation with a given initial condition. The solution involved selection of the portion of $\ln|y-1|$ that includes the initial condition. Part (c) asked for the limit of the solution from part (b) as $x \to \infty$.

Sample: 5A
Score: 9

The student earned all 9 points.

Sample: 5B
Score: 6

The student earned 6 points: 1 point in part (a), 5 points in part (b), and no points in part (c). In part (a) the student did not earn the first point but earned the second point. In part (b) the student earned the point for separation of variables, both points for antidifferentiating, and the points for the constant of integration and use of the initial condition. However, the student presents the equation for the tangent line at the initial condition point rather than solving for the function that satisfies the differential equation.

Sample: 5C
Score: 4

The student earned 4 points: 1 point in part (a), 3 points in part (b), and no points in part (c). In part (a) the student earned the first point. Since the slopes on the x-axis are not correct, the student did not earn the second point. The student also has slope segments drawn on the y-axis. In part (b) the student earned points for the separation of variables, the constant of integration, and use of the initial condition. The student did not earn any points for antidifferentiation. Since the antiderivative with respect to y does not include absolute values, the student is not able to correctly solve for y.

Question 6

Let f be the function given by $f(x) = \dfrac{\ln x}{x}$ for all $x > 0$. The derivative of f is given by $f'(x) = \dfrac{1 - \ln x}{x^2}$.

(a) Write an equation for the line tangent to the graph of f at $x = e^2$.

(b) Find the x-coordinate of the critical point of f. Determine whether this point is a relative minimum, a relative maximum, or neither for the function f. Justify your answer.

(c) The graph of the function f has exactly one point of inflection. Find the x-coordinate of this point.

(d) Find $\lim\limits_{x \to 0^+} f(x)$.

(a) $f\left(e^2\right) = \dfrac{\ln e^2}{e^2} = \dfrac{2}{e^2}$, $f'\left(e^2\right) = \dfrac{1 - \ln e^2}{\left(e^2\right)^2} = -\dfrac{1}{e^4}$

An equation for the tangent line is $y = \dfrac{2}{e^2} - \dfrac{1}{e^4}\left(x - e^2\right)$.

$2 : \begin{cases} 1 : f\left(e^2\right) \text{ and } f'\left(e^2\right) \\ 1 : \text{answer} \end{cases}$

(b) $f'(x) = 0$ when $x = e$. The function f has a relative maximum at $x = e$ because $f'(x)$ changes from positive to negative at $x = e$.

$3 : \begin{cases} 1 : x = e \\ 1 : \text{relative maximum} \\ 1 : \text{justification} \end{cases}$

(c) $f''(x) = \dfrac{-\frac{1}{x}x^2 - (1 - \ln x)2x}{x^4} = \dfrac{-3 + 2\ln x}{x^3}$ for all $x > 0$

$f''(x) = 0$ when $-3 + 2\ln x = 0$

$x = e^{3/2}$

The graph of f has a point of inflection at $x = e^{3/2}$ because $f''(x)$ changes sign at $x = e^{3/2}$.

$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer} \end{cases}$

(d) $\lim\limits_{x \to 0^+} \dfrac{\ln x}{x} = -\infty$ or Does Not Exist

1 : answer

 AB

NO CALCULATOR ALLOWED

6A₁

Do not write beyond this border.

Work for problem 6(a)

$f(e^2) = \frac{\ln(e^2)}{e^2} = \frac{2}{e^2}$

$y - \frac{2}{e^2} = \left(\frac{1-\ln(e^2)}{e^4}\right)(x - e^2)$

$y - \frac{2}{e^2} = -\frac{1}{e^4}(x - e^2)$

Work for problem 6(b)

$f'(x) = 0 = \frac{1 - \ln x}{x^2}$

$1 - \ln x = 0 \qquad x^2 = 0$

$\ln x = 1 \qquad \cancel{x = 0}$ not in domain

$x = e$

$f'(x)$ + − e

Since $f'(x)$ changes from positive to negative @ $x = e$, this point is a relative maximum.

Continue problem 6 on page 15.

6A2

Work for problem 6(c)

$$f''(x) = \frac{(x^2)(1-\ln x)' - (1-\ln x)(x^2)'}{x^4}$$

$$= \frac{-x - 2x(1-\ln x)}{x^4}$$

$$0 = \frac{-x - 2x + 2x\ln x}{x^4}$$

$$0 = -3x + 2x\ln x = x(2\ln x - 3)$$

$x = 0$ $\quad -3 + 2\ln x = 0$

$\ln x = \frac{3}{2}$

$\boxed{x = e^{3/2}}$

$f''(x)$ $-$ $+$

$e^{3/2}$

There is an inflection pt @ $x = e^{3/2}$ because $f''(x)$ changes from $- \to +$ there.

Work for problem 6(d)

$$\lim_{x \to 0^+} \frac{\ln x}{x} = \frac{\ln(\text{really small pos \#})}{\text{really small pos \#}}$$

$$= \frac{\text{really Big neg \#}}{\text{really small pos \#}}$$

$= \text{really, really big neg \#}$

$\therefore \lim_{x \to 0^+} f(x) = -\infty$

GO ON TO THE NEXT PAGE.

6B1

Do not write beyond this border.

Work for problem 6(a)

$F(x) = \frac{\ln x}{x}$ $F'(x) = \frac{1-\ln x}{x^2}$

$F(e^2) = \frac{\ln e^2}{e^2}$ $F'(e^2) = \frac{1-\ln e^2}{(e^2)^2}$

$= \frac{2}{e^2}$

$= \frac{1-2}{e^4}$

$= -\frac{1}{e^4}$

$y = mx + b$

$\frac{2}{e^2} = -\frac{1}{e^{\cancel{4}}}(\cancel{e^2}) + b$

$\frac{2}{e^2} = -\frac{1}{e^2} + b$

$b = \frac{3}{e^2}$

$\boxed{y = -\frac{1}{e^4}x + \frac{3}{e^2}}$

Work for problem 6(b)

crit. point of $F \Rightarrow F'(x) = 0$

$\ln x^2$
$2\ln x$

$\frac{1-\ln x}{x^2} = 0$ und@ 0

$x = e$

und + +

0 e

critical pt. $= \boxed{e}$ $\boxed{\text{neither}}$

Continue problem 6 on page 15.

6B2

Work for problem 6(c)

$$f''(x) = \frac{\left(-\frac{1}{\cancel{x}}\right)\cancel{x^2} - (1-\ln x)2x}{x^4}$$

$$= \frac{-x - 2x(1-\ln x)}{x^4}$$

$$= \frac{-x-2x+2x\ln x}{x^4}$$

$$= \frac{-3+2\ln x}{x^3} = 0$$

$$2\ln x = 3$$

$$\ln x = 3/2$$

$$\boxed{x = e^{3/2}}$$

Work for problem 6(d)

L'hospitals

$$\lim_{x\to 0^+} \frac{\ln x}{x} = \frac{\text{und}}{\text{und}}$$

$$\frac{\frac{1}{x}}{1} = \frac{1}{x} \qquad \boxed{\infty}$$

Do not write beyond this border.

Do not write beyond this border.

NO CALCULATOR ALLOWED

6C1

Do not write beyond this border.

Work for problem 6(a)

$f(x)=\frac{\ln x}{x}$ $\quad f'(x)=\frac{1-\ln x}{x^2}$

$f(e^2)=\frac{\ln e^2}{e^2}=\frac{2}{e^2}$

$(e^2, \frac{2}{e^2})$

$y-\frac{2}{e^2}=\left(\frac{1-\ln x}{x^2}\right)(x-e^2)$ $\quad \boxed{y=\left(\frac{1-\ln x}{x}\right)-e^2\left(\frac{1-\ln x}{x^2}\right)+\frac{2}{e^2}}$

Work for problem 6(b)

$f'(x)=(1-\ln x)(x^{-2})=0$

$\frac{(1-\ln x)}{x^2}=0$

u + + + + 0 - - - - -

0 e

$1-\ln x=0$

$1=\ln x$

$\boxed{x=e}$

u + + + 0 - - -

0 e

this point is a relative maximum because the function is increasing $0<x<e$ and decreasing $e<x<\infty$.

Continue problem 6 on page 15.

NO CALCULATOR ALLOWED

6C2

Do not write beyond this border.

Work for problem 6(c)

$f'(x) = (1-\ln x)(x^{-2})$.

$f''(x) = (1-\ln x)(-2x^{-3}) + (x^{-2})\left(-\frac{1}{x}\right)$

$\frac{-2(1-\ln x)}{x^3} - \frac{1}{x^3}\left(\frac{-1}{x}\right) = \frac{-2(1-\ln x)-1}{x^3} = 0$

$-2 + 2\ln x - 1 = 0$

$2\ln x - 1 = 0$

$\ln x = \frac{1}{2}$

$\boxed{x = e^{\frac{1}{2}}}$

Work for problem 6(d)

$f(x) = \frac{\ln x}{x}$ L'Hopitals rule $\rightarrow \lim_{x \to 0^+} \frac{1}{x} = \frac{1}{0}$ $\boxed{\text{undefined}}$

Do not write beyond this border.

GO ON TO THE NEXT PAGE.

Question 6

Overview

This problem presented students with a function f defined by $f(x) = \frac{\ln x}{x}$ for $x > 0$, together with a formula for $f'(x)$. Part (a) asked for an equation of the line tangent to the graph of f at $x = e^2$. In part (b) students needed to solve $f'(x) = 0$ and determine the character of this critical point from the supplied $f'(x)$. In part (c) students had to demonstrate skill with the quotient rule to obtain a formula for $f''(x)$ and solve $f''(x) = 0$ to find the x-coordinate of what was promised to be the only point of inflection for the graph of f. Part (d) tested students' knowledge of properties of $\ln x$ to determine the limit of $f(x)$ as $x \to 0^+$.

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 2 points in part (a), 1 point in part (b), 3 points in part (c), and no points in part (d). In part (a) the student identifies the value of the function and the derivative at $x = e^2$ and correctly writes a tangent line equation. In part (b) the student correctly identifies $x = e$ but classifies it as neither a minimum nor a maximum so only earned the first point. In part (c) the student gives the correct second derivative and correctly solves the equation. In part (d) the student's answer is not correct.

Sample: 6C
Score: 4

The student earned 4 points: no points in part (a), 2 points in part (b), 2 points in part (c), and no points in part (d). In part (a) the student does not identify the value of the derivative at $x = e^2$ and does not write an equation of the tangent line. In part (b) the student correctly identifies $x = e$ and classifies $x = e$ as a maximum but does not give a justification. The student gives only the definition of a maximum. In part (c) the student earned 2 points by correctly applying the product rule to find the second derivative. The student makes an arithmetic error in solving the equation and did not earn the third point. In part (d) the student has an acceptable answer but did not earn the point because of the reference to L'Hospital's Rule. This limit is not a candidate for L'Hospital's Rule.